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Singular Limits of Spatially Inhomogeneous Convection-reaction-diffusion Equations

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Abstract We study convection-reaction-diffusion equations which have spatially inhomogeneous steady states. When the coefficients of the reaction terms are much larger than the diffusion coefficients, sharp interfaces appear between two stable inhomogeneous steady states. By using the method of matched asymptotic expansions, we derive the equations of the motion of such interfaces, which depend on the mean curvature of the interfaces and depend on the inhomogeneous coefficients locally.

Keywords Convection-reaction-diffusion equations · Singular limits · Interface equations · Matched asymptotic expansions

1 Introduction

It is known that some classes of nonlinear diffusion equations give rise to sharp interfaces (or internal layers) when the coefficients of the reaction terms are much larger than the diffusion coefficients. And the motion of such interfaces is often driven by their mean curvatures.

To name only a few, Chen [3] gave a rigorous proof on the generation and propagation of interfaces for

$$u_t = \Delta u + \frac{1}{\varepsilon^2} f(u), \quad x \in \mathbb{R}^N,\ t > 0,$$

where $\varepsilon > 0$ is a small parameter and $f(u)$ is a bistable nonlinear term. More precisely, f has exactly three zeros: $z_1 < z_2 < z_3$, $f'(z_1) < 0$, $f'(z_2) > 0$, $f'(z_3) < 0$ and $\int_{z_1}^{z_3} f(s)ds = 0$. A typical example is $f(u) = u - u^3$. Chen [3] showed that, in time scale $t = O(1)$, the motion of the interface (interface equation) is

$$V = -(N-1)\kappa,$$

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where V is the normal velocity of the interface and κ is the mean curvature of it. In [1] and [9], the authors studied the singular limit of a reaction diffusion equation with a spatially inhomogeneous reaction term:

$$u_t = \Delta u + \frac{1}{\varepsilon^2} h^2(x) f(u), \quad x \in \mathbb{R}^N, \ t > 0, \tag{1.1}$$

where $h(x) > 0$, f is bistable as above. They obtained the interface equation like the following (1.3). In [7], we considered a p-Laplacian reaction-diffusion equation:

$$\frac{1}{\varepsilon^{p-2}} u_t = \operatorname{div}(a^p(x)|\nabla u|^{p-2}\nabla u) + \frac{1}{\varepsilon^p} h^p(x) f(u), \quad x \in \mathbb{R}^N, \ t > 0, \tag{1.2}$$

where $p > 1$ is a constant, $h(x) > 0$, $a(x) > \delta > 0$ and f is bistable as above. We showed that the interface equation is

$$V = -g\left[(N-1)a^2 h^{p-2}\kappa + a^2 h^{p-3}\frac{\partial h}{\partial n} + (p-1)ah^{p-2}\frac{\partial a}{\partial n}\right], \tag{1.3}$$

where g is a constant depending on p and f, n is the unit normal vector to the interface (outward, for closed interface). Both of the interface equations deriving from (1.1) and (1.2) involve drift terms arising from h.

Note that all of the above listed reaction diffusion equations have three homogeneous steady states z_i $(i = 1, 2, 3)$ (z_1 and z_3 are stable and z_2 is unstable). In this paper, we consider convection-reaction-diffusion equations with spatially inhomogeneous steady states. First we consider

$$u_t = a(x)\Delta u + c(x)\nabla b(x) \cdot \nabla u + \frac{1}{\varepsilon^2} R(x, u), \quad x \in \mathbb{R}^N, \ t > 0, \tag{1.4}$$

where a, b and c are smooth functions, $a(x) > \delta > 0$ and

$$R(x, u) = H(x)F(x, u) + \varepsilon h(x) f(x, u),$$

where H, h, F, f are smooth functions, H is positive. F and f satisfy the following Assumption 1.1:

Assumption 1.1 There exist bounded, smooth functions $z_1(x)$, $z_2(x)$ and $z_3(x)$ with $\inf[z_2(x) - z_1(x)] > 0$, $\inf[z_3(x) - z_2(x)] > 0$ such that

$$F(x, z_i(x)) \equiv 0 \quad (i = 1, 2, 3), \qquad F(x, y) \neq 0 \quad \text{if } y \neq z_i(x),$$
$$F_u(x, z_1(x)) < 0, \qquad F_u(x, z_2(x)) > 0, \qquad F_u(x, z_3(x)) < 0,$$
$$\int_{z_1(x)}^{z_3(x)} F(x, s)ds \equiv 0, \quad \text{and} \quad f(x, z_1(x)) \equiv f(x, z_3(x)) \equiv 0.$$

When $F(x, u) = F(u)$, the above assumption means that F is a typical bistable nonlinearity with double-well potential of equal well-depth (cf. [1] and [9]). An example of inhomogeneous R is

$$R(x, u) = -H(x)(u - z + \omega)(u - z + \varepsilon\tilde{h}(x))(u - z - \omega),$$

where $z = z(x)$ and $\omega = \omega(x)$ are bounded functions and $\inf \omega(x) > 0$.

The main difference between (1.4) and (1.1) (or (1.2)) is the following: (1.1) and (1.2) has spatially homogeneous steady states z_i $(i = 1, 2, 3)$, but (1.4) does not have. In fact, a formal approach in the next section shows that (1.4) has three steady states $\bar{u}_i(x) \approx z_i(x)$ $(i = 1, 2, 3)$, $\bar{u}_1$ and $\bar{u}_3$ are stable, $\bar{u}_2$ is unstable. When $\varepsilon \ll 1$, sharp interfaces appear between $\bar{u}_1$ and $\bar{u}_3$ in time scale $t = O(\varepsilon^2)$, and the interfaces propagate in time scale $t = O(1)$. By using the matched asymptotic expansions, we will give the equation of the motion of such an interface, which indicates that the motion of the interface depends on the mean curvature and on the local information of the inhomogeneous coefficients (see (3.10) below).

We will also study a system of convection-reaction-diffusion equations

$$\begin{cases} u_t = a(x,v)\Delta u + c(x,v)\nabla(b(x,v)) \cdot \nabla u + \varepsilon^{-2}R(x,u,v), \\ v_t = \varepsilon^{-1}[\alpha(x,u)\Delta v + u - \beta v], \end{cases} \tag{1.5}$$

for $x \in \mathbb{R}^N$, $t > 0$, where $a(x,v)$, $b(x,v)$, $c(x,v)$, $\alpha(x,u)$ are smooth functions with $a(x,u) > \delta > 0$ and $\alpha(x,u) > \delta > 0$. β is a constant and

$$R(x,u,v) = H(x,v)F(x,u) + \varepsilon h(x,v)f(x,u),$$

where H, h are smooth functions and H is positive, F and f satisfy Assumption 1.1. This equation is a spatially inhomogeneous version of FitzHugh-Nagumo equations (cf. [1] and [9]). By using the matched asymptotic expansions, we also derive the interface equation which depends on the mean curvature and on the spatially inhomogeneous coefficients (see (4.8) below).

In Sect. 2, we discuss the spatially inhomogeneous steady states. In Sect. 3, we use matched asymptotic expansions to derive the interface equation from (1.4). In Sect. 4, we derive the interface equation from (1.5).

2 Inhomogeneous Steady States

One can easily see that $z_i(x)$ are not exactly steady states of (1.4). Substituting

$$\bar{u}_i(x) = z_i(x) + \varepsilon\bar{u}_{i1}(x) + \varepsilon^2\bar{u}_{i2}(x) + \varepsilon^3\bar{u}_{i3}(x) + \cdots \quad (i = 1, 2, 3) \tag{2.1}$$

into (1.4) and collecting the $\varepsilon^{-2}, \varepsilon^{-1}, \varepsilon^0, \ldots$ terms we have

$$HF(x, z_i(x)) = 0, \tag{2.2}$$

$$HF_u(x, z_i(x))\bar{u}_{i1} + hf(x, z_i(x)) = 0, \tag{2.3}$$

$$a\Delta z_i + c\nabla b \cdot \nabla z_i + HF_u(x, z_i)\bar{u}_{i2} + HF_{uu}(x, z_i)\bar{u}_{i1}^2/2 + hf_u(x, z_i)\bar{u}_{i1} = 0, \tag{2.4}$$

$$\ldots.$$

Then one can easily obtain $\bar{u}_{ij}$ $(i = 1, 2, 3,\ j = 1, 2, \ldots)$. Especially, $\bar{u}_{11} = \bar{u}_{31} = 0$. In such a way, we formally obtain the steady states of (1.4), which are inhomogeneous functions given by (2.1).

Since

$$F_u(x, z_1(x)) < 0, \qquad F_u(x, z_2(x)) > 0, \qquad F_u(x, z_3(x)) < 0,$$

and since $\bar{u}_i(x) \approx z_i(x)$ $(i = 1, 2, 3)$, we know that when ε is sufficiently small, $\bar{u}_1$ and $\bar{u}_3$ are stable steady states, and $\bar{u}_2$ is unstable steady state.

Substituting (2.1) and

$$\bar{v}_i(x) = \bar{v}_{i0}(x) + \varepsilon \bar{v}_{i1}(x) + \varepsilon^2 \bar{v}_{i2}(x) + \cdots \quad (i = 1, 2, 3)$$

into (1.5) and collecting the $\varepsilon^{-2}, \varepsilon^{-1}, \varepsilon^0, \ldots$ terms one can also obtain, formally, three steady states $(\bar{u}_i, \bar{v}_i)$ $(i = 1, 2, 3)$ with $\bar{u}_i(x) \approx z_i(x)$ and $\bar{v}_i(x) \approx \bar{v}_{i0}(x)$. Here, $\bar{v}_{i0}$ $(i = 1, 2, 3)$ are given by

$$\alpha(x, z_i)\Delta \bar{v}_{i0} - \beta \bar{v}_{i0} = -z_i(x).$$

Hence these steady states are also inhomogeneous.

3 Interface Equation of (1.4)

In this section, we present a formal derivation of the equation of motion of interface for (1.4). The technique is based on matched asymptotic expansions using the so-called signed distance function, which can be found in [5] and [9], etc.

3.1 Matched Asymptotic Expansions

Let u^ε be a solution of (1.4) and Γ^ε be the interface

$$\Gamma^\varepsilon = \bigcup_{t \geq 0} (\Gamma_t^\varepsilon \times \{t\}),$$

where $\Gamma_t^\varepsilon = \{x \in \mathbb{R}^N \mid u^\varepsilon(x, t) = \bar{u}_2(x)\}$. Hereafter, we assume that the interface Γ^ε is smooth and that Γ_t^ε is a smooth closed hypersurface in $\mathbb{R}^N$ without boundaries for each $t \geq 0$. We denote by Ω_t^ε the bounded domain in $\mathbb{R}^N$ enclosed by Γ_t^ε.

Let $d^\varepsilon(x, t)$ be the signed distance function to Γ^ε defined by

$$d^\varepsilon(x, t) = \begin{cases} \mathrm{dist}(x, \Gamma_t^\varepsilon), & x \in \mathbb{R}^N \setminus \overline{\Omega_t^\varepsilon}, \\ -\mathrm{dist}(x, \Gamma_t^\varepsilon), & x \in \Omega_t^\varepsilon. \end{cases} \tag{3.1}$$

We remark that $d^\varepsilon = 0$ on Γ^ε and $|\nabla d^\varepsilon| = 1$. We assume that d^ε has the expansion

$$d^\varepsilon(x, t) = d_0(x, t) + \varepsilon d_1(x, t) + \varepsilon^2 d_2(x, t) + \cdots$$

and denote

$$\Gamma_t = \{x \in \mathbb{R}^N \mid d_0(x, t) = 0\}, \qquad \Omega_t = \{x \in \mathbb{R}^N \mid d_0(x, t) < 0\},$$
$$\Gamma = \bigcup_{t \geq 0} (\Gamma_t \times \{t\}), \qquad Q_0 = \bigcup_{t \geq 0} ((\mathbb{R}^N \setminus \overline{\Omega_t}) \times \{t\}), \qquad Q_1 = \bigcup_{t \geq 0} (\Omega_t \times \{t\}).$$

Roughly speaking, Γ_t represents the position of the interface at time t in the limit as $\varepsilon \to 0$, while Ω_t represents the region inside Γ_t.

We also assume that the solution u^ε has the expansions

$$u^\varepsilon(x, t) = u_0(x, t) + \varepsilon u_1(x, t) + \varepsilon^2 u_2(x, t) + \cdots \tag{3.2}$$

away from the interface Γ^ε (the outer expansion) and

$$u^\varepsilon(x, t) = U_0(\xi, x, t) + \varepsilon U_1(\xi, x, t) + \varepsilon^2 U_2(\xi, x, t) + \cdots \tag{3.3}$$

near Γ^ε (the inner expansion), where $\xi = d^\varepsilon(x,t)/\varepsilon$. The stretched space variable ξ gives exactly the right spatial scaling to describe the sharp transition between the regions $\{u \approx \bar{u}_1(x) \approx z_1(x)\}$ and $\{u \approx \bar{u}_3(x) \approx z_3(x)\}$. Since $u^\varepsilon = \bar{u}_2(x)$ on Γ^ε, we normalize U_k in such a way that $U_0(0,x,t) = z_2(x)$, $U_k(0,x,t) = \bar{u}_{2k}$ $(k = 1, 2, \dots)$ for all (x,t) near Γ^ε (normalization conditions). To make the inner and outer expansions consistent, we require that

$$U_k(+\infty, x, t) = u_k^+(x,t) \quad \text{if } x \in (\mathbb{R}^N \backslash \overline{\Omega_t}) \cup \Gamma_t, \tag{3.4}$$

$$U_k(-\infty, x, t) = u_k^-(x,t) \quad \text{if } x \in \Omega_t \cup \Gamma_t, \tag{3.5}$$

for all (x,t) near Γ and all $k \geq 0$ (matching conditions), where u_k^+ and u_k^- respectively denote the terms of outer expansion (3.2) in the region Q_0 and the region Q_1. In particular, if $x \in \Gamma_t$, then one has to take into account both of the conditions (3.4) and (3.5).

3.2 Motion of Interface for (1.4)

Substituting the outer expansion (3.2) into (1.4) and collecting the ε^{-2} and ε^{-1} terms respectively, we have

$$HF(x, u_0) = 0, \qquad HF_u(x, u_0)u_1 + hf(x, u_0) = 0,$$

in $Q_0 \cup Q_1$. Hence we have $u_0 = z_1(x)$, or $u_0 = z_2(x)$, or $u_0 = z_3(x)$ in $Q_0 \cup Q_1$. Since we are studying interfaces between the region $\{u \approx z_1(x)\}$ and $\{u \approx z_3(x)\}$, we have either $u_0(x,t) = z_1(x)$ in Q_1 and $u_0(x) = z_3(x)$ in Q_0 or the other way around. As both cases are treated similarly, we will assume the former throughout this section. Therefore $f(x, u_0) \equiv 0$ and from the second equation we get $u_1(x,t) = 0$ in $Q_0 \cup Q_1$.

Now, using (3.3) we have

$$\begin{aligned} u_t^\varepsilon &= \frac{1}{\varepsilon} U_{0\xi} \cdot d_{0t} + (U_{0\xi} \cdot d_{1t} + U_{0t} + U_{1\xi} \cdot d_{0t}) + \cdots, \\ \nabla u^\varepsilon &= \frac{1}{\varepsilon} U_{0\xi} \cdot \nabla d^\varepsilon + \nabla U_0 + U_{1\xi} \cdot \nabla d^\varepsilon + \varepsilon \nabla U_1 + \varepsilon U_{2\xi} \cdot \nabla d^\varepsilon + \cdots, \\ \Delta u^\varepsilon &= \frac{1}{\varepsilon^2} U_{0\xi\xi} + \frac{1}{\varepsilon}(2\nabla U_{0\xi} \cdot \nabla d^\varepsilon + U_{0\xi} \Delta d^\varepsilon + U_{1\xi\xi}) + \cdots, \end{aligned}$$

where ∇ and Δ denote the derivatives with respect to x (regarding ξ, t as parameters). Substituting them into (1.4) and collecting the ε^{-2} and ε^{-1} terms we have

$$a(x)U_{0\xi\xi} + H(x)F(x, U_0) = 0, \tag{3.6}$$

and

$$a(x)U_{1\xi\xi} + HF_u(x, U_0)U_1 = A(\xi, x, t), \tag{3.7}$$

where

$$A(\xi, x, t) = U_{0\xi} d_{0t} - 2a \nabla U_{0\xi} \cdot \nabla d_0 - aU_{0\xi} \Delta d_0 - cU_{0\xi} \nabla b \cdot \nabla d_0 - hf(x, U_0).$$

For any given (x,t), combining with the matching conditions and normalization condition, we consider

$$\begin{cases} a(x)U_{0\xi\xi} + H(x)F(x, U_0) = 0, \\ U_0(-\infty, x, t) = z_1(x), \quad U_0(0, x, t) = z_2(x), \quad U_0(+\infty, x, t) = z_3(x). \end{cases} \tag{3.8}$$

Since (x,t) is fixed, it is known that (3.8) has a unique solution $U_0(\xi,x,t)=U_0(\xi,x)$ (cf. [1] and [6]). Moreover,

(P1) $|U_0(\xi,x)-z_1(x)|=O(e^{-\delta|\xi|})$ as $\xi\to-\infty$, $|U_0(\xi,x)-z_3(x)|=O(e^{-\delta|\xi|})$ as $\xi\to+\infty$ for some $\delta>0$.
(P2) $U_0(\xi,x)$ depends on x smoothly.
(P3) $U_{0\xi}(\xi,x)=\sqrt{2[W(x,U_0)-W(x,z_1(x))]}>0$ for any $U_0(\xi,x)\in\mathbb{R}\backslash\{z_1(x),z_3(x)\}$, where $W(x,U)=-\frac{H(x)}{a(x)}\int_0^U F(x,s)ds$.

To solve (3.7), we use a Fredholm type lemma (see for example, Lemma 1.2.2 in [1], or Lemma 4.1 in [2], or Lemma 2.1 in [9]), which says that (3.7) with normalization condition $U_1(0,x,t)=\bar{u}_{21}(x)$ has a solution in $L^\infty(\mathbb{R})$ if and only if

$$\int_{\mathbb{R}} A(\xi,x,t)U_{0\xi}(\xi,x)d\xi=0,$$

that is,

$$(d_{0t}-a\Delta d_0-c\nabla b\cdot\nabla d_0)p(x)-a\nabla d_0\cdot\nabla p(x)-hq(x)=0,$$

where

$$p(x)=\int_{\mathbb{R}} U_{0\xi}^2 d\xi,\qquad q(x)=\int_{\mathbb{R}} f(x,U_0)U_{0\xi}d\xi=\int_{z_1(x)}^{z_3(x)} f(x,s)ds. \tag{3.9}$$

It is easily seen that ∇d_0 coincides with the outward normal unit vector n to the hypersurface Γ_t and $-d_{0t}(x,t)=V$, where V is the normal velocity of the interface Γ_t. It is also known that the mean curvature κ of the interface is $\frac{\Delta d_0}{N-1}$. Thus we obtain the following interface equation

$$V=-(N-1)a(x)\kappa-c(x)\frac{\partial b(x)}{\partial n}-a(x)\frac{\partial(\log p(x))}{\partial n}-h(x)\frac{q(x)}{p(x)}. \tag{3.10}$$

4 Interface Equation of (1.5)

Substituting (3.2) and

$$v^\varepsilon(x,t)=v_0(x,t)+\varepsilon v_1(x,t)+\varepsilon^2 v_2(x,t)+\cdots \tag{4.1}$$

into (1.5) and collecting the ε^{-2} and ε^{-1} terms, we have

$$H(x,v_0)F(x,u_0)=0, \tag{4.2}$$

$$H(x,v_0)F_u(x,u_0)u_1+h(x,v_0)f(x,u_0)+H_v(x,v_0)F(x,u_0)v_1=0, \tag{4.3}$$

$$\alpha(x,u_0)\Delta v_0+u_0-\beta v_0=0, \tag{4.4}$$

in $Q_0\cup Q_1$. By (4.2) we have $u_0(x,t)=z_1(x)$ in Q_1 and $u_0(x)=z_3(x)$ in Q_0 (or the other way around). Hence in (4.3), $f(x,u_0)\equiv F(x,u_0)\equiv 0$, and so $u_1(x,t)=0$ in $Q_0\cup Q_1$ independently of v_0. Moreover, by (4.4) we have

$$\begin{cases}\alpha(x,z_1(x))\Delta v_0-\beta v_0=-z_1(x), & x\in\Omega_t,\\ \alpha(x,z_3(x))\Delta v_0-\beta v_0=-z_3(x), & x\in\mathbb{R}^N\backslash\overline{\Omega_t}.\end{cases} \tag{4.5}$$

Substituting (3.3) and (4.1) into the first equation of (1.5) and collecting the ε^{-2} and ε^{-1} terms, we get

$$a(x,v_0)U_{0\xi\xi} + H(x,v_0)F(x,U_0) = 0, \tag{4.6}$$

$$a(x,v_0)U_{1\xi\xi} + H(x,v_0)F_u(x,U_0)U_1 = B(\xi,x,t), \tag{4.7}$$

where

$$B(\xi,x,t) = U_{0\xi}d_{0t} - 2a(x,v_0)\nabla U_{0\xi}\cdot\nabla d_0 - a(x,v_0)U_{0\xi}\Delta d_0 \\ - U_{0\xi}c(x,v_0)\nabla(b(x,v_0))\cdot\nabla d_0 - H_v(x,v_0)F(x,U_0)v_1 - h(x,v_0)f(x,U_0).$$

For any fixed (x,t), solving (4.6) with the matching conditions and the normalization condition:

$$U_0(-\infty,x,t) = z_1(x), \qquad U_0(0,x,t) = z_2(x), \qquad U_0(\infty,x,t) = z_3(x),$$

we have a unique solution $U_0(\xi,x,t) = U_0(\xi,x)$ which satisfies the properties (P1)–(P3) in the previous section.

Using Fredholm type lemma to consider the solvability for (4.7), we find that (4.7) is solvable in $L^\infty(\mathbb{R})$ if and only if

$$\int_{\mathbb{R}} B(\xi,x,t)U_{0\xi}d\xi = 0.$$

Noting $\int_{z_1(x)}^{z_3(x)} F(x,s)ds \equiv 0$ we have

$$d_{0t} - a(x,v_0)\Delta d_0 - c(x,v_0)\nabla(b(x,v_0))\cdot\nabla d_0 - a(x,v_0)\nabla d_0\cdot\frac{\nabla p(x)}{p(x)} - h(x,v_0)\frac{q(x)}{p(x)} = 0,$$

where p and q are defined by (3.9) with new U_0 we obtained in this section. Therefore, we obtain the interface equation

$$V = -(N-1)a(x,v_0)\kappa - c(x,v_0)\frac{\partial b(x,v_0)}{\partial n} - a(x,v_0)\frac{\partial(\log p(x))}{\partial n} - h(x,v_0)\frac{q(x)}{p(x)}, \tag{4.8}$$

where v_0 is given by (4.5).

5 Some Remarks

Remark 5.1 In (3.10), the equation contains spatially inhomogeneous coefficients a, b, c, h, p and q, where p and q are defined by U_0 and f. From the definition of U_0 we find that U_0 in fact depends only on a, H, F, z_1, z_2, z_3 locally. Hence we can say that the interface equation (mean curvature flow equation) depends on local information of spatially inhomogeneous steady states. Similar discussion is true for (4.8).

Remark 5.2 Since p depends only on a, H, F, z_1, z_2 and z_3, in (3.10) and in (4.8) when we choose b and c such that $c\nabla b + a\nabla(\log p) = 0$, then we obtain an interface equation with the form

$$V = -(N-1)\varphi(x)\kappa + \psi(x).$$

In case $N = 2$ it has been studied systematically in [4]. Travelling wave solutions of a special case: $V = -\kappa + C$ (where C is a constant) has also been studied recently by some authors (see for example [8]).

Remark 5.3 Our approach in this paper is also valid for discussing the singular limit of gas discharge equations [10] (system of three-component reaction diffusion equations):

$$\begin{cases} u_t = a(x, v, w)\Delta u + c(x, v, w)\nabla(b(x, v, w)) \cdot \nabla u + \varepsilon^{-2} R(x, u, v, w), \\ v_t = \varepsilon^{-1}[\alpha(x, u)\Delta v + u - \beta v], \quad x \in \mathbb{R}^N, \ t > 0, \\ w_t = \varepsilon^{-1}[\lambda(x, u)\Delta w + u - \mu w], \end{cases}$$

where $R(x, u, v, w) = H(x, v, w)F(x, u) + \varepsilon h(x, v, w) f(x, u)$, and F and f satisfy Assumption 1.1. Similar results as for (1.5) can be obtained.

Remark 5.4 Using the matched asymptotic expansions, one can even consider the interface equation for convection-reaction-diffusion equations with time-dependent coefficients

$$u_t = a(x, t)\Delta u + c(x, t)\nabla b(x, t) \cdot \nabla u + \varepsilon^{-2} R(x, t, u).$$

In some cases, though the equation has no steady states, sharp interfaces still generate and propagate according to the mean curvature.

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References

1. Alfaro, M.: Convection-reaction-diffusion systems and interface dynamics. PhD. Thesis, University of Paris XI (2006)
2. Alikakos, N.D., Bates, P.W., Chen, X.: Convergence of the Cahn-Hilliard equation to the Hele-Shaw model. Arch. Ration. Mech. Anal. **128**, 165–205 (1994)
3. Chen, X.: Generation and propagation of interfaces for reaction-diffusion equations. J. Differ. Equ. **96**, 116–141 (1992)
4. Chou, K.-S., Zhu, X.-P.: The Curve Shortening Problem. Chapman & Hall/CRC, Boca Raton (2001)
5. Fife, P.C.: Dynamics of Internal Layers and Diffusive Interfaces. CBMS-NSF Regional Conf. Ser. in Appl. Math. SIAM, Philadelphia (1988)
6. Fife, P.C., McLeod, J.B.: The approach of solutions of nonlinear diffusion equations to travelling front solutions. Arch. Ration. Mech. Anal. **65**, 335–361 (1977)
7. Lou, B.: Singular limit of a p-Laplacian reaction-diffusion equation with a spatially inhomogeneous reaction term. J. Stat. Phys. **110**(1/2), 377–383 (2003)
8. Matano, H., Nakamura, K.-I., Lou, B.: Periodic traveling waves in a two-dimensional cylinder with saw-toothed boundary and their homogenization limit. Netw. Heterog. Media **1**(4), 537–568 (2006)
9. Nakamura, K.-I., Matano, H., Hilhorst, D., Schatzle, R.: Singular limit of a reaction-diffusion equations with a spatially inhomogeneous reaction term. J. Stat. Phys. **95**(5,6), 1165–1185 (1999)
10. Nishiura, Y., Teramoto, T., Ueda, K.-I.: Scattering and separators in dissipative systems. Phys. Rev. E **67**, 056210 (2003)